

maintained), sharply decreasing with a decline in the load. This region disappears only at the instant the plate stops.

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ELASTOPLASTIC STRAIN OF THIN PLATES AND SHELLS UNDER LINEAR HARDENING AND AN IDEAL BAUSCHINGER EFFECT

G. V. Ivanov

UDC 539.3

The elastoplastic strain of thin plates and shells is considered in the case when the elongation and shears are small compared with unity, the hardening is linear, the Bauschinger effect is ideal, and the stresses and strains are related by equations [1, 2]. In solving problems numerically by using the equations [1, 2], it is necessary to evaluate integrals over the plate (shell) thickness and thus to store and process, respectively, information about the stresses, the residual microstresses, and the nature of the strain at the sites over the plate (shell) thickness during the solution. Analogously to the case of ideal elastoplastic strain of plates and shells [3], approximate equations which contain no stresses and relate the strain directly to the forces and moments are formulated below in correspondence to the equations in [1, 2]. The need to evaluate integrals over the plate (shell) thickness drops out in solving problems by using these equations, which simplifies the solution. Numerical experiments performed for a number of strain paths of the shell element exhibit satisfactory agreement of the approximate equations with the equations of [1, 2].

§1. Let us use a Lagrange coordinate system, orthogonal in the unstrained state, to write the equations. For small elongations and shears, the system under consideration can be considered orthogonal in the strain state as well. The strain and stress tensor components are related in the case of elastoplastic strain with linear hardening and an ideal Bauschinger effect (Fig. 1) by the equations [1, 2]

$$\begin{aligned}
 e_{ij} &= (1 + \nu) \sigma_{ij} - 3\nu \delta_{ij} \sigma + \gamma \eta_{ij}, & \eta_{ij} &= \lambda s'_{ij}; \\
 \lambda &= 0, & \text{if } 3T^2 < 1 \text{ or } 3T^2 = 1, T' < 0; \\
 \lambda &> 0, & \text{if } 3T^2 = 1, T' = 0;
 \end{aligned}$$

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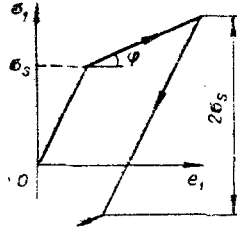


Fig. 1

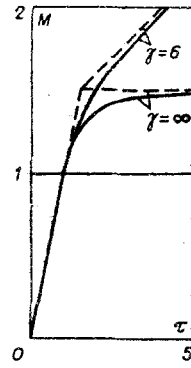


Fig. 2

$$\begin{aligned} s'_{ij} &= \sigma'_{ij} - \eta_{ij}, & \sigma'_{ij} &= \sigma_{ij} - \delta_{ij}\sigma, & \sigma &= \frac{1}{3} \delta_{ij} \sigma_{ij}, \\ T^2 &= \frac{1}{2} s'_{ij} s'_{ij}, & \gamma &= \frac{3}{2} (E/E' - 1), & i, j &= 1, 2, 3, \end{aligned} \quad (1.1)$$

where σ_{ij} , e_{ij} are the stress and strain referred, respectively, to σ_S and σ_S/E , σ_S is the yield point under uniaxial tension (see Fig. 1), E is Young's modulus, the point denotes differentiation with respect to the loading parameter, ν is the Poisson ratio, and $E' = \tan \varphi$ is the tangent modulus of the uniaxial tension diagram (see Fig. 1).

Let us assume that the stress state of the shell is planar ($\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$) but the strain corresponds to the Kirchhoff hypothesis. Let us use the notation

$$\begin{aligned} e_{\alpha\beta} &= \varepsilon_{\alpha\beta} + 2\zeta \kappa_{\alpha\beta}, & \zeta &= 2z/h, & s_{\alpha\beta} &= s'_{\alpha\beta} + \delta_{\alpha\beta} s, \\ s &= -s'_{33} = \delta_{\alpha\beta} s'_{\alpha\beta} = \frac{1}{3} \delta_{\alpha\beta} s_{\alpha\beta}, & \alpha, \beta &= 1, 2, \end{aligned} \quad (1.2)$$

z is a coordinate measured from the middle surface along the normal to it, and h is the shell thickness. Evidently,

$$\sigma_{\alpha\beta} = s_{\alpha\beta} + \eta_{\alpha\beta} + \delta_{\alpha\beta} \eta, \quad \eta = \delta_{\alpha\beta} \eta_{\alpha\beta}. \quad (1.3)$$

§2. Let us assume that a shell element is completely (over the whole thickness) deformed plastically and the elastic strain rates are negligibly small compared to the plastic strain rates. In this case

$$\begin{aligned} \dot{e}_{\alpha\beta} + 2\zeta \dot{\kappa}_{\alpha\beta} &= \gamma \dot{\eta}_{\alpha\beta}, & \dot{\eta}_{\alpha\beta} &= \lambda \dot{s}_{\alpha\beta}, \\ 3T^2 &= \frac{3}{2} s'_{\alpha\beta} s'_{\alpha\beta} = \frac{3}{2} (s'_{\alpha\beta} s'_{\alpha\beta} + s^2) = 1 \end{aligned} \quad (2.1)$$

and, therefore,

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \gamma q_{\alpha\beta}, & \kappa_{\alpha\beta} &= \frac{3}{4} \gamma \theta_{\alpha\beta}, & q_{\alpha\beta} &= \frac{1}{2} \int_{-1}^1 \eta_{\alpha\beta} d\zeta, & \theta_{\alpha\beta} &= \int_{-1}^1 \eta_{\alpha\beta} \zeta d\zeta, \\ p_{\alpha\beta} \dot{e}_{\alpha\beta} + \mu_{\alpha\beta} \dot{\kappa}_{\alpha\beta} &= \gamma \dot{\Lambda}, & p_{\alpha\beta} &= \frac{1}{2} \int_{-1}^1 s_{\alpha\beta} d\zeta, & \mu_{\alpha\beta} &= \int_{-1}^1 s_{\alpha\beta} \zeta d\zeta, \\ 6\Lambda\gamma &= 2\gamma \int_{-1}^1 \lambda d\zeta = \sqrt{6} \int_{-1}^1 [(\dot{e}_{\alpha\beta} + 2\zeta \dot{\kappa}_{\alpha\beta})(\dot{e}_{\alpha\beta} + 2\zeta \dot{\kappa}_{\alpha\beta}) + \\ &+ [\delta_{\alpha\beta} (\dot{e}_{\alpha\beta} + 2\zeta \dot{\kappa}_{\alpha\beta})]^2]^{1/2} d\zeta, & p_{\alpha\beta} &= \gamma \partial \Lambda / \partial \dot{e}_{\alpha\beta}, & \mu_{\alpha\beta} &= \gamma \partial \Lambda / \partial \dot{\kappa}_{\alpha\beta}. \end{aligned} \quad (2.2)$$

According to (2.2), the six quantities $p_{\alpha\beta}$, $\mu_{\alpha\beta}$ are functions of five arguments - ratios of strain rates. Hence, the same dependence holds between $p_{\alpha\beta}$, $\mu_{\alpha\beta}$ as between the forces and moments in the limit state of ideally plastic shells [3, 4]. Let us approximate it by the equation [3, 4]

$$\begin{aligned} f &= 1, & f &= Q_t + \frac{1}{2} Q_m - (Q_t Q_m - Q_{tm}^2)/4(Q_t + 0.48 Q_{tm}) + \\ &+ \frac{1}{2} \sqrt{Q_m^2 + 4Q_{tm}^2}, & Q_t &= \frac{3}{2} p_{\alpha\beta} p'_{\alpha\beta}, & Q_m &= \frac{3}{2} \mu_{\alpha\beta} \mu'_{\alpha\beta}, & Q_{tm} &= \frac{3}{2} p_{\alpha\beta} \mu'_{\alpha\beta}, \end{aligned}$$

$$p'_{\alpha\beta} = p_{\alpha\beta} - \delta_{\alpha\beta} p, \quad p = \frac{1}{3} \delta_{\alpha\beta} p_{\alpha\beta}, \quad \mu'_{\alpha\beta} = \mu_{\alpha\beta} - \delta_{\alpha\beta} \mu, \quad \mu = \frac{1}{3} \delta_{\alpha\beta} \mu_{\alpha\beta}. \quad (2.3)$$

It has been shown in [4] that the dependence (2.3) differs insignificantly from the exact value.

Using (2.3), let us replace (2.1) by the approximations

$$\begin{aligned} \dot{\varepsilon}_{\alpha\beta} &= \gamma \dot{q}_{\alpha\beta}, \quad \dot{\kappa}_{\alpha\beta} = \frac{3}{4} \gamma \dot{\theta}_{\alpha\beta}, \quad f = 1, \\ \dot{q}_{\alpha\beta} &= \varphi \partial f / \partial p_{\alpha\beta}, \quad \dot{\theta}_{\alpha\beta} = \frac{4}{3} \varphi \partial f / \partial \mu_{\alpha\beta}. \end{aligned} \quad (2.4)$$

Let $t_{\alpha\beta}$, $m_{\alpha\beta}$ denote the forces and moments in the shell:

$$t_{\alpha\beta} = \frac{1}{2} \int_{-1}^1 \sigma_{\alpha\beta} d\xi, \quad m_{\alpha\beta} = \int_{-1}^1 \sigma_{\alpha\beta} \xi d\xi.$$

Using (1.3), we find

$$\begin{aligned} t_{\alpha\beta} &= p_{\alpha\beta} + q_{\alpha\beta} + \delta_{\alpha\beta} q, \quad q = \delta_{\alpha\beta} q_{\alpha\beta}, \\ m_{\alpha\beta} &= \mu_{\alpha\beta} + \theta_{\alpha\beta} + \delta_{\alpha\beta} \theta, \quad \theta = \delta_{\alpha\beta} \theta_{\alpha\beta}. \end{aligned} \quad (2.5)$$

Equations (2.4) and (2.5) form an approximate system of equations relating the strain rates $\dot{\varepsilon}_{\alpha\beta}$, $\dot{\kappa}_{\alpha\beta}$ to the forces $t_{\alpha\beta}$ and moments $m_{\alpha\beta}$ in conformity with the case when the shell element is completely (over the whole thickness) deformed plastically and the elastic strain rates are negligibly small compared with the plastic strain rates.

§3. In the general case of deformation of a shell element, we assume that for $f < 1$ as well as $f = 1$, $f' < 0$, the plastic strain rates over the whole thickness of the element are zero (the element is deformed elastically), while for $f = 1$, $f' = 0$ the plastic strain rates are determined by (2.4). Correspondingly,

$$\begin{aligned} \dot{\varepsilon}_{\alpha\beta} &= (1 + \nu) \dot{t}_{\alpha\beta} - 3\nu \delta_{\alpha\beta} \dot{t} + \gamma \dot{q}_{\alpha\beta}, \quad \dot{q}_{\alpha\beta} = c \varphi \partial f / \partial p_{\alpha\beta}, \\ \dot{\kappa}_{\alpha\beta} &= \frac{3}{4} [(1 + \nu) \dot{m}_{\alpha\beta} - 3\nu \delta_{\alpha\beta} \dot{m} + \gamma \dot{\theta}_{\alpha\beta}], \quad \dot{\theta}_{\alpha\beta} = \frac{4}{3} c \varphi \partial f / \partial \mu_{\alpha\beta}, \\ c &= \begin{cases} 0, & \text{if } f < 1 \text{ or } f = 1, f' < 0, \\ 1, & \text{if } f = 1, f' = 0, \end{cases} \\ \dot{t} &= \frac{1}{3} \delta_{\alpha\beta} \dot{t}_{\alpha\beta}, \quad \dot{m} = \frac{1}{3} \delta_{\alpha\beta} \dot{m}_{\alpha\beta}. \end{aligned} \quad (3.1)$$

Supplementing (3.1) by the relationships (2.5), we obtain an approximate system of elastoplastic plate and shell strain equations with linear hardening and an ideal Bauschinger effect. It contains no stresses and relates the strain rates $\dot{\varepsilon}_{\alpha\beta}$, $\dot{\kappa}_{\alpha\beta}$ directly to the forces $t_{\alpha\beta}$ and moments $m_{\alpha\beta}$. From (2.5) and (3.1) we find

$$\begin{aligned} (1 - \nu^2) \dot{p}_{\alpha\beta} &= (1 - \nu) \dot{\varepsilon}_{\alpha\beta} + \nu \delta_{\alpha\beta} \dot{\varepsilon} - (4 - 5\nu + \gamma) \nu \delta_{\alpha\beta} \dot{q} - \nu(1 + \nu + \gamma) \dot{q}_{\alpha\beta}, \\ (1 - \nu^2) \dot{\mu}_{\alpha\beta} &= \frac{4}{3} [(1 - \nu) \dot{\kappa}_{\alpha\beta} + \nu \delta_{\alpha\beta} \dot{\kappa}] - (4 - 5\nu + \gamma) \nu \delta_{\alpha\beta} \dot{\theta} - \nu(1 + \nu + \gamma) \dot{\theta}_{\alpha\beta}, \\ \dot{\varepsilon} &= \delta_{\alpha\beta} \dot{\varepsilon}_{\alpha\beta}, \quad \dot{\kappa} = \delta_{\alpha\beta} \dot{\kappa}_{\alpha\beta} \end{aligned}$$

and, therefore,

$$\begin{aligned} (1 - \nu^2) f' &= \Omega' - c \varphi S, \\ \Omega' &= [(1 - \nu) \dot{\varepsilon}_{\alpha\beta} + \nu \delta_{\alpha\beta} \dot{\varepsilon}] \partial f / \partial p_{\alpha\beta} + \frac{4}{3} [(1 - \nu) \dot{\kappa}_{\alpha\beta} + \nu \delta_{\alpha\beta} \dot{\kappa}] \partial f / \partial \mu_{\alpha\beta}, \\ S &= \nu \{ (4 - 5\nu + \gamma) [(\partial f / \partial p_{11} + \partial f / \partial p_{22})^2 + (4/3)(\partial f / \partial \mu_{11} + \partial f / \partial \mu_{22})^2] + \\ &\quad + (1 + \nu + \gamma) [\partial f / \partial p_{\alpha\beta} \cdot \partial f / \partial p_{\alpha\beta} + (4/3) \partial f / \partial \mu_{\alpha\beta} \cdot \partial f / \partial \mu_{\alpha\beta}] \}. \end{aligned}$$

Hence, the function φ in (3.1) can be written as

$$\varphi = \Omega' / S,$$

and the conditions for c in (3.1) can be replaced by the conditions

$$c = \begin{cases} 0, & \text{if } f < 1 \text{ or } f = 1, \Omega' \leq 0, \\ 1, & \text{if } f = 1, \Omega' > 0. \end{cases}$$

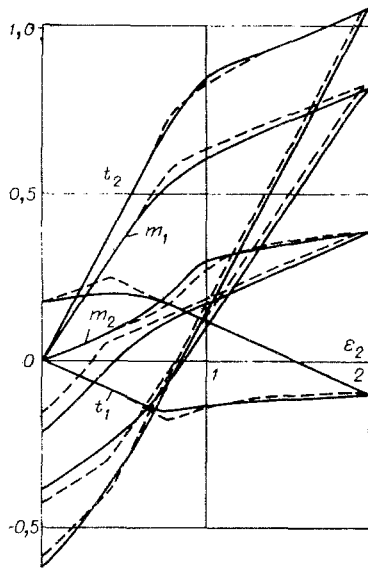


Fig. 3

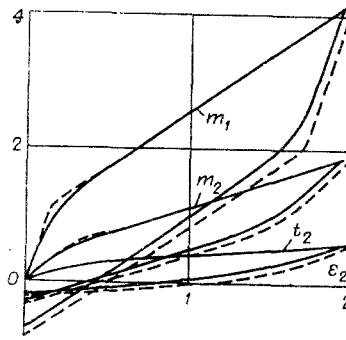


Fig. 4

In solving problems by using (3.1), it is convenient to evaluate the function f and its derivatives by means of the formulas

$$\begin{aligned}
 f &= aQ_t + 2bQ_{tm} + dQ_m, \quad \partial f / \partial p_{\alpha\beta} = 3(a p'_{\alpha\beta} + b \mu'_{\alpha\beta}), \\
 \partial f / \partial \mu_{\alpha\beta} &= 3(b p'_{\alpha\beta} + d \mu'_{\alpha\beta}), \quad a = 1 - (Q_m^2 + 0.48Q_m^2) N_1^2, \\
 N_1 &= (Q_t + 0.48Q_m)^{-1}, \quad b = Q_{tm} \left(N_2 + \frac{1}{4} N_1 \right), \quad N_2 = (Q_m^2 + 4Q_{tm}^2)^{-1/2}, \\
 d &= \frac{1}{2} (1 + Q_m N_2) - (Q_t^2 + 0.48Q_{tm}^2) N_1^2.
 \end{aligned}$$

§4. In the case $\gamma = 0$ the shell element is deformed elastically, and (3.1) and (2.5) correspond exactly to (1.1)-(1.3). In the case $\gamma = \infty$, Eqs. (1.1)-(1.3) go over into the equations of ideal elastoplastic shell strain [3]. The correspondence between (3.1) and (2.5) and (1.1)-(1.3) is satisfactory in this case [3].

To compare the results from using (3.1) and (2.5) with the results from (1.1)-(1.3), let us consider the bending of a shell element for a proportionate change in the curvature,

$$e_{11} = 2\zeta k_1 t, \quad e_{22} = 2\zeta k_2 t, \quad e_{12} = 0,$$

for $0 < \gamma < \infty$, where k_1 and k_2 are constants, and t is the strain parameter. For simplicity let us set $\nu = 1/2$.

In this case, the strain according to (1.1)-(1.3) will be elastic,

$$\sigma_{11} = \frac{4}{3} \zeta (2k_1 + k_2), \quad \sigma_{22} = \frac{4}{3} \zeta (2k_2 + k_1), \quad (4.1)$$

for $t < t_0$ as well as for $t \geq t_0$, $|\zeta| < \zeta_0$

$$t_0 = \sqrt{3} [16(k_1^2 + k_1 k_2 + k_2^2)]^{-1/2}, \quad \zeta_0 = t_0/t.$$

The deformation will be plastic,

$$\begin{aligned}
 \sigma_{11} &= 4\zeta (2k_1 + k_2) / (3 + 2\gamma), \\
 \sigma_{22} &= 4\zeta (2k_2 + k_1) / (3 + 2\gamma),
 \end{aligned} \quad (4.2)$$

for $t \geq t_0$, $|\zeta| \geq \zeta_0$. Let us use the notation

$$\begin{aligned}
 M &= 9m_{11} [8(2k_1 + k_2)t_0]^{-1} = 9m_{22} [8(2k_2 + k_1)t_0]^{-1}, \\
 \tau &= t/t_0.
 \end{aligned}$$

From (4.1) and (4.2) we find

$$M = \tau, \quad \tau \leq 1; \quad M = [\tau + \gamma - (1/3)\gamma\tau^2] [1 + (2/3)\gamma]^{-1}, \quad \tau \geq 1. \quad (4.3)$$

According to (3.1) and (2.5),

$$M = \tau, \quad \tau \leq 3/2; \quad M = (\tau + \gamma) [1 + (2/3)\gamma]^{-1}, \quad \tau \geq 3/2. \quad (4.4)$$

The difference between the results of using (4.3) and (4.4) for $0 < \gamma < \infty$ evidently does not exceed the difference for $\gamma = \infty$. Results of calculations using (4.3) (solid lines) and (4.4) (dashed lines) are presented in Fig. 2 for $\gamma = 6$, $\gamma = \infty$.

Results of calculating the forces and moments by means of (1.1)-(1.3) (solid lines) and (3.1) and (2.5) (dashed lines) are presented in Figs. 3 and 4 for a shell element deformed according to the law

$$\begin{aligned} \dot{\varepsilon}_{12} = \dot{\varkappa}_{12} = \dot{\varkappa}_{22} = 0, \quad \dot{\varkappa}_{11} = \chi \dot{\varepsilon}_{22}, \quad \dot{\varepsilon}_{11} = -(1/2) \dot{\varepsilon}_{22}, \\ \dot{\varepsilon}_{22} = 1, \quad 0 < t \leq t_1; \quad \dot{\varepsilon}_{22} = -1, \quad t_1 < t \leq 2t_1, \end{aligned}$$

where the point denotes differentiation with respect to t . The results in Fig. 3 correspond to $\chi = 0.5$ and in Fig. 4 to $\chi = 5$. The calculation was performed for $\nu = 0.3$, $\gamma = 6$, $t_1 = 2$. In evaluating $t_{\alpha\beta}$, $m_{\alpha\beta}$ by means of (1.1)-(1.3) the integrals were replaced by Simpson quadratures with 21 sites. The computation procedure is analogous to that elucidated in [3].

The results in Figs. 2-4, the calculations for other shell element strain paths, and the comparison with the results in [3] all show that (3.1) and (2.5) correspond satisfactorily to (1.1)-(1.3), the difference in the results for $\gamma < \infty$ not exceeding this difference in the case of ideal elastoplastic shell strain [3].

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SINGULAR SOLUTIONS OF EQUATIONS OF SHALLOW SHELLS FOR A CONCENTRATED TANGENTIAL LOAD

V. P. Ol'shanskii

UDC 539.3

As we know [1, 2], in the case of action of concentrated loads the solutions of the shell equations have a singular character. These solutions have been set up by various methods primarily for a normal concentrated force. An attempt to obtain the fundamental solutions for a tangential force lead to very cumbersome results [3]. Below, by the method of Fourier integral transforms, it was possible to obtain more compact solutions in the form of power and trigonometric series. As an addition to the well-known results in the analysis of singularities of the stress state in the vicinity of a concentrated source of radius r , it is shown that in addition to the tangential forces increasing as r^{-1} for $r \rightarrow 0$, one of the shear forces also has a weaker singularity of logarithmic form. Asymptotic expressions of the behavior of the fundamental solutions for small values of the argument are given.

The analysis of the elastic local stress state is carried out on the basis of the equations of the theory of thin, shallow, isotropic shells. The solution of these equations by means of the two-dimensional Fourier transform, which is expounded in detail in [3], gives the following values of the components of internal force quantities:

$$\begin{aligned} t_1 = \frac{-2X}{1-\nu} \left[\frac{1-\nu}{2} A_1 + \frac{2-\nu-\nu^2}{2} A_2 + \left(a_1 + \frac{\lambda+\nu}{1-\nu^2} b^4 a_5 \right) A_5 + \right. \\ \left. + \left(a_2 - \nu a_3 + \frac{\lambda+\nu}{1-\nu^2} b^4 a_4 \right) A_6 \right], \quad t_2 = \frac{-2X}{1-\nu} \left[\nu \frac{1-\nu}{2} A_1 + \frac{\nu-1}{2} A_2 + \right. \\ \left. + \left(\nu a_1 + \frac{1+\lambda\nu}{1-\nu^2} b^4 a_5 \right) A_5 + \left(\nu a_2 - a_3 + \frac{1+\lambda\nu}{1-\nu^2} b^4 a_4 \right) A_6 \right], \end{aligned}$$

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